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Heuristics as an alternative to variational calculus for optimization of a class of thermal insulation systems

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Abstract

This article reports an alternative treatment in lieu of the principle of variational calculus for a certain class of optimization problems. In particular, the optimum distribution of insulating material on one side of a flat plate for minimum heat transfer is sought when the other side is exposed to a laminar forced convection. Both conjugate and non-conjugate formulations of the problem are conceived and closed form solutions are presented. Interestingly, optimized insulation profile exhibits a category of equipartition principle in some macroscopic domain. Expression for minimum heat transfer is a function of Biot number in non-conjugate analysis of the model. Contrastingly, the non-dimensional group $\overline{J}h_{\mathrm{L}}$ is the characteristic parameter for conjugate formulation. Finally, Bejan's method of intersecting asymptotes is employed to find an order of magnitude for a ceiling value of the wall material. With some scale factor, a range $0 < \overline{J}_{\max} \leqslant 1.506Pr^{-1/3}$ for the representative material volume can be ascertained, beyond which the optimization exercise reduces to a trivial one and traditional constant thickness profile becomes a recognized design.

Keywords: Equipartition; Heuristics; Intersecting asymptotes; Synthetic constraint; Thermal insulation; Variational calculus

1. Introduction

In conventional formulation of heat transfer between a stream of fluid and flat plate, boundary conditions are normally stipulated at the solid–liquid interface i.e., at the top of the plate as shown in Fig. 1. However in a large number of applications the temperature at the bottom surface of the plate is either specified or can be estimated. If the plate is of negligible thickness or has a high thermal conductivity, the temperature drop between the top and bottom surface can be neglected and the problem is solved purely in convective heat transfer regime. When thickness of the plate is not negligible or even varying along its length and the thermal conductivity of the wall material is poor, the boundary conditions at the bottom of the plate is to be considered and the whole problem is to be reformulated as a conductive–convective one. This is a fundamental mathematical challenge imposed by the design criterion of thermal insulating systems.

The present discussion has the reference of conductive–convective heat transfer along a flat plate of variable thickness and is due to Lim et al. [1]. The goal of the work was the optimal distribution of a limited quantity of insulating material on the backside of a convectively cooled flat plate. In the first part of their paper the authors assumed a boundary layer type variation

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Nomenclature

A	non-dimensional parameter, Eq. (35)
b	non-dimensional taper parameter, Eq. (33)
B	non-dimensional parameter, Eq. (35)
C	non-dimensional parameter, Eq. (35)
Bi	Biot number, Eq. (6)
Ec	Eckert number
$f(\eta)$	function obtained from Blasius solution
$h(x)$	local convective heat transfer coefficient
h_L	heat transfer coefficient at the extreme downstream
J	dimensionless group, Eq. (17)
$\bar{J}$	non-dimensional physical parameter, Eq. (16)
k_f	thermal conductivity of fluid
k_w	thermal conductivity of insulating material
L	length of the flat plate
n	exponent in heat transfer co-efficient relation, Eq. (3)
Pr	Prandtl number
q'	heat transfer rate per unit length, Eqs. (5) and (21)
q''	local heat flux, Eq. (4)
$\bar{R}$	total conductive and convective resistance, Eq. (7)
Re_x	local Reynolds number, $U_\infty x/v$
Re_L	Reynolds number at the extreme downstream, $U_\infty L/v$
$\dot{S}_{gen}$	entropy generation rate, Eq. (42)
$\underline{\dot{S}}_{gen}$	uniform entropy generation rate, Eq. (43)
t	variable wall thickness
$\bar{t}$	length-based averaged wall thickness, Eq. (2a)
T_0	temperature at the bottom surface of the plate
T_∞	temperature of the free stream
ΔT	constant thermal potential difference, Eq. (1)
u	velocity component along the flat plate
U_∞	free stream velocity
v	velocity component normal to the flat plate
x	coordinate along the flat plate, Figs. 1 and 2
y	coordinate perpendicular to the flat plate, Figs. 1 and 2

Greek symbols

α	dummy variable, Eq. (18)
β	dummy variable, Eq. (18)
χ	arbitrary design variable, Eq. (44)
$\bar{\chi}$	total magnitude of all dissipative forces, Eq. (44)
η	similarity variable, Eq. (12)
θ	non-dimensionalized fluid temperature, Eq. (12)
λ	Lagrange multiplier, Eq. (23)
v	kinematic viscosity
ξ	dimensionless coordinate, x/L
Φ	aggregate integral, Eq. (23)

Subscripts

constant	uniform wall thickness distribution, $t = \bar{t}$
i	number of competing dissipating mechanisms
m	dummy variable, Eq. (44)
max	maximum
min	minimum with conjugate formulation, Eq. (29)
*min	minimum with non-conjugate formulation, Eq. (10)
opt	optimum wall thickness distribution, Eq. (28)
taper	tapered wall thickness distribution, Eq. (33)
*	optimum wall thickness distribution, Eq. (9)

of convective heat transfer coefficient and determined the total thermal resistance as a series combination contributed by conduction and convection. Accounting for the constraint of insulation volume they could cast the optimization problem in Euler–Lagrange form and obtained an analytical solution. The second part of their paper solves the same problem of forced convection using the conjugate heat transfer condition without assuming any heat transfer coefficient before hand. They provided a general formulation for convective cooling of a flat plate with lagging of arbitrary thickness on the other surface. However, they failed to extend the calculus of variation to find out the optimum profile of insulation in this case. Instead of considering the optimum non-linear profile thickness they numerically determined the total rate of heat loss for a linearly varying tapered shape and showed it to be less than that of a plate with constant thickness of insulation.

In the present investigation the above problem has been revisited. Firstly, it has been shown that the result deduced by variational calculus can be obtained by a simple heuristic logic grounded on the physics of the problem. Secondly, the heuristic logic is not only applicable for the first part of the problem [1] with known variation of heat transfer coefficient but can also be employed for the second part where the estimation of heat transfer is based on a truly conjugate formulation. Further, an approximate bound of the insulation volume

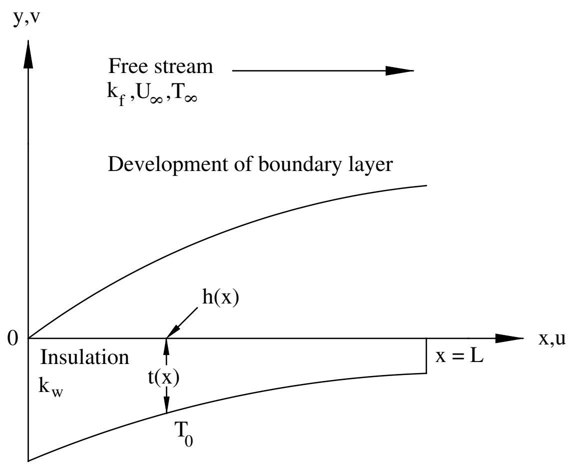

Fig. 1. Flat plate with variable thickness at bottom and convection on top.

has been provided by Bejan's method of intersecting asymptotes for any meaningful optimization of the insulation design. An analytical treatment has been extended for the design of tapered insulation profile. Finally, it has been argued that present problem demonstrates a category of equipartition.

2. The physical model

A flat plate with variable thickness and finite length is considered as shown in Fig. 1. The bottom of the plate is exposed to an environment with high convective heat transfer coefficient such that the temperature of the surface remains practically uniform at T_0. Top and the flat side of the plate is in thermal communication with a different flow characteristic, (k_f, U_∞, T_∞). Total inventory of the wall material is fixed. It is desirable to seek an optimal distribution of the wall material to achieve minimum heat transfer from the plate [1]. This is a fundamental optimization problem encountered in the design of thermal insulating systems.

In the present analysis, we assume that the driving potential for heat transfer remains piecewise constant i.e.,

$$T_\infty - T_0 = \Delta T \quad \text{(Constant)}. \tag{1}$$

The material volume per unit length is a constant and can be expressed in terms of average thickness as

$$\bar{t} = \frac{1}{L}\int_0^L t(x)\mathrm{d}x \tag{2a}$$

or

$$\int_0^1 \frac{t(\xi)}{\bar{t}}\,\mathrm{d}\xi = 1, \quad \text{with } \xi = \frac{x}{L}. \tag{2b}$$

3. Optimization with assumed variation of heat transfer coefficient

In a forced convection heat transfer from a flat plate it is legitimate to assume a power law variation of heat transfer coefficient along the direction of the flow in the form [1]

$$h = h_L \xi^{-n}, \tag{3}$$

where h_L is the lowest value of heat transfer coefficient at the extreme downstream $x = L$ and n is an exponent.

Local heat flux q'' driven by temperature potential ΔT can be expressed considering convective and conductive resistances in series as

$$q'' = \frac{\Delta T}{\frac{t(x)}{k_w} + \frac{1}{h(x)}}. \tag{4}$$

Eq. (4) can be integrated for the entire length of the plate invoking Eq. (3) to obtain total heat transfer rate q' per unit length perpendicular to the plane in the dimensionless from as

$$\frac{q'}{k_w L \Delta T/\bar{t}} = \int_0^1 \frac{\mathrm{d}\xi}{\frac{t}{\bar{t}} + \frac{\xi^n}{Bi}}, \tag{5}$$

where

$$Bi = \frac{h_L \bar{t}}{k_w}. \tag{6}$$

Lim et al. [1] constructed an aggregate integral combining Eqs. (5) and (2b) through a Lagrange multiplier. Finally, the Euler–Lagrange equation of the integral was solved to find out the optimum thickness of insulation.

In the present work the problem has been attacked by a simple heuristic logic. As the goal is to reduce the heat loss from the total length of the plate it is instructive to provide the maximum thickness of insulation where the coefficient of convective heat transfer is maximum. In other words, one should equip the highest conductive resistance where the convective resistance is minimum. This exercise should be carried out for the entire plate length under the constraint of limited insulation material. Logically this exercise can terminate only when uniform total thermal resistance (conductive plus convective) exists throughout the length of the plate. Mathematically, this translates into the equation with the stipulation that the denominator of the integrand in Eq. (5) is constant i.e.,

$$\frac{t}{\bar{t}} + \frac{\xi^n}{Bi} = \overline{R} \quad \text{(Constant)}. \tag{7}$$

A schematic for this heuristic logic is graphically supplemented in Fig. 2.

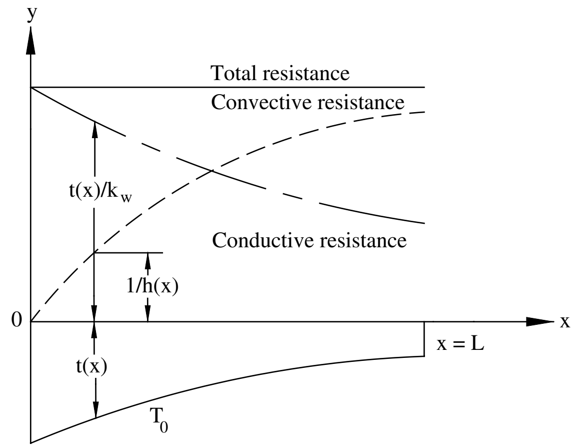

Fig. 2. Conceptual basis of heuristic model.

Providing an expression for $t(\xi)/\bar{t}$ in Eq. (2b) from Eq. (7), we obtain another expression for total resistance as

$$\frac{1}{(n+1)Bi}+1=\overline{R}. \tag{8}$$

Eliminating $\overline{R}$ from Eq. (7) and Eq. (8) leads to the functional form of optimal distribution of insulation thickness t_* as

$$\frac{t_*}{\bar{t}}=1+\frac{1}{Bi}\left(\frac{1}{n+1}-\xi^n\right), \tag{9}$$

where Biot number, Bi is defined in Eq. (6).

Minimum heat transfer rate from Eq. (5) invoking Eq. (9) in the denominator of the integrand is obtained in non-dimensional form as

$$\frac{q'_{*\min}}{k_w L\Delta T/\bar{t}}=\frac{Bi}{Bi+(n+1)^{-1}}. \tag{10}$$

Eqs. (9) and (10) are the important results for the optimum allocation of insulation and were obtained employing calculus of variation in open literature [1].

4. Optimization with unknown variation of convective heat transfer coefficient

In actual practice neither the variation of temperature at the top surface of the plate nor convective heat transfer coefficient is known a priori [1]. Rather it is to be determined from one of conjugate convective–conductive formulation of the problem. Neglecting dissipation ($Ec \ll 1$), boundary layer energy equation can be written in the form [2]

$$\frac{d^2\theta}{d\eta^2}+\frac{1}{2}Pr\,f\frac{d\theta}{d\eta}=0, \tag{11}$$

with the definitions

$$\eta=\frac{y}{\sqrt{\nu x/U_\infty}}=\frac{y}{x}Re_x^{1/2},$$

$$\theta(\eta)=\frac{T_f-T_0}{T_\infty-T_0}\quad\text{and}\quad\frac{df}{d\eta}=\frac{u}{U_\infty}. \tag{12}$$

The similarity function $f(\eta)$ is obtained from the momentum equation of Blasius form [3]. Free stream boundary condition by definition reads as

$$\theta\to 1\quad\text{at } \eta\to\infty\ (y\to\infty). \tag{13}$$

Considering maximum thickness of the wall is much smaller than its length, longitudinal conduction through the insulating material can be neglected. Thus, the conjugate boundary condition can be modeled at the interface $y=0$ as

$$k_f\left(\frac{\partial T_f}{\partial y}\right)_{0^+}=k_w\left(\frac{T_w-T_0}{t(x)}\right)_{0^-} \tag{14a}$$

and

$$T_f=T_w. \tag{14b}$$

The non-dimensional version of these two boundary conditions, Eqs. (14a) and (14b), become

$$J\frac{\partial\theta}{\partial\eta}=\theta\quad\text{at } \eta=0, \tag{15}$$

where

$$\bar{J}=\frac{k_f}{k_w}\frac{\bar{t}}{L}Re_L^{1/2} \tag{16}$$

and

$$J=\bar{J}\frac{t}{\bar{t}}\xi^{-1/2}. \tag{17}$$

The dimensionless number $\bar{J}$ is in general a function of x except for some special functional form of $t(x)$. The quantity $\bar{J}\to 0$ represents Pohlhausen limit that is for the isothermal plate with negligible wall thickness.

Eq. (11) can be integrated in a straightforward manner using the relation, Eq. (17) and boundary conditions Eqs. (13) and (15) to yield

$$\theta'(0)=\left\{\bar{J}\frac{t}{\bar{t}}\xi^{-1/2}+\int_0^\infty \exp\left[-\frac{Pr}{2}\int_0^\beta f(\alpha)d\alpha\right]d\beta\right\}^{-1}. \tag{18}$$

But the improper integral in the denominator of Eq. (18) is well known in the literature [4] and for $Pr > 0.5$ it is most accurately correlated as

$$\int_0^\infty \exp\left[-\frac{Pr}{2}\int_0^\beta f(\alpha)d\alpha\right]d\beta=\left(0.332Pr^{1/3}\right)^{-1}. \tag{19}$$

Our concern is to calculate overall heat transfer rate through the entire length of the plate using the relation

$$q' = \int_0^L k_f \left(\frac{\partial T}{\partial y}\right)_{y=0^+} \mathrm{d}x$$
$$= \Delta T k_f Re_L^{1/2} \int_0^1 \theta'(0)\xi^{-1/2}\,\mathrm{d}\xi \tag{20}$$

etc. Invoking Eqs. (18) and (19) into Eq. (20) a non-dimensional equation for heat transfer is resulted as

$$\frac{q'}{k_w L \Delta T/\bar{t}} = \int_0^1 \frac{\mathrm{d}\xi}{\frac{t}{\bar{t}} + \frac{\xi^n}{\bar{J}h_L}}, \tag{21}$$

with

$$n = 1/2 \quad \text{and} \quad h_L = 0.332 Pr^{1/3} \text{ in particular.} \tag{22}$$

It may be noted that Eq. (21) has a form exactly equivalent to that of Eq. (5). Therefore, the same heuristic logic can be extended and the optimum variation of insulation thickness can be determined readily.

It may further be noted that even for this purely conjugate heat transfer situation, one can exploit the calculus of variation to obtain the optimum profile for insulation thickness. Employing Eqs. (21) and (2b) one may formulate a problem of unconstrained optimization with the introduction of a Lagrange multiplier as

$$\Phi = \int_0^1 \left(\frac{1}{\frac{t(\xi)}{\bar{t}} + \frac{\xi^n}{\bar{J}h_L}} + \lambda \frac{t(\xi)}{\bar{t}} \right) \mathrm{d}\xi = \int_0^1 F\,\mathrm{d}\xi, \tag{23}$$

where the factor λ is a Lagrange multiplier and F is the shorthand for the integrand. The optimal thickness is the solution of the following Euler–Lagrange equation

$$\frac{\partial F}{\partial t} - \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{\partial F}{\partial(\mathrm{d}t/\mathrm{d}\xi)}\right] = 0. \tag{24}$$

Since, the integrand in Eq. (23) is independent of the slope of the profile, Eq. (24) takes a simple look

$$\frac{\partial F}{\partial t} = 0. \tag{25}$$

The resulting expression for optimal thickness distribution involving Lagrange multiplier stands as

$$\frac{t(\xi)}{\bar{t}} = (\lambda)^{-1/2} - \frac{\xi^n}{\bar{J}h_L}. \tag{26}$$

From the volume constraint, Eq. (2b) we obtain another expression for the parameter $(\lambda)^{-1/2}$ as

$$(\lambda)^{-1/2} = 1 + \frac{1}{(n+1)\bar{J}h_L}. \tag{27}$$

Combining Eqs. (26) and (27) we conclude with the expression

$$\frac{t_{opt}}{\bar{t}} = 1 + \frac{1}{\bar{J}h_L}\left(\frac{1}{n+1} - \xi^n\right). \tag{28}$$

Employing this profile shape, non-dimensionalized minimum heat transfer from Eq. (21) reads as

$$\frac{q'_{min}}{k_w L \Delta T/\bar{t}} = \frac{\bar{J}h_L}{\bar{J}h_L + (n+1)^{-1}}. \tag{29}$$

One may check that the heuristic logic also provides the same result as given in Eqs. (28) and (29).

5. Bounds of insulation volume

It is implied that optimization for minimum heat transfer is a worthy endeavor only when amount of insulating material falls into a limit. To bracket this limit one can integrate Eq. (21) for two different extreme conditions.

When there is acute scarcity of insulating material passing to the lower limit $\bar{J} \to 0$ we have

$$\operatorname*{Lt}_{\bar{J}\to 0}\left(\frac{q'}{k_w L \Delta T/\bar{t}}\right) = \operatorname*{Lt}_{\bar{J}\to 0}\left[\bar{J}\int_0^1 \frac{\mathrm{d}\xi}{\bar{J}\frac{t}{\bar{t}} + \frac{\xi^{1/2}}{0.332Pr^{1/3}}}\right]$$
$$= 0.664 Pr^{1/3}\bar{J}. \tag{30}$$

This is the classical Pohlhausen solution with no thickness of the wall or having high conductivity of the wall material. On the other hand, for over abundance of insulating material, optimization for the profile shape is trivial, that is $t \to \bar{t}$. Passing to the higher limit $\bar{J} \to \infty$ we obtain

$$\operatorname*{Lt}_{\bar{J}\to \infty}\left(\frac{q'}{k_w L \Delta T/\bar{t}}\right) = \operatorname*{Lt}_{\bar{J}\to \infty}\left[\bar{J}\int_0^1 \frac{\mathrm{d}\xi}{\bar{J}\frac{t}{\bar{t}} + \frac{\xi^{1/2}}{0.332Pr^{1/3}}}\right] = 1. \tag{31}$$

Now, we are positioned to fix an upper ceiling for the insulating material employing Bejan's method of intersecting asymptotes [5,6]. Elimination of q' terms between Eqs. (30) and (31) yields

$$\bar{J}_{max} = 1.506 Pr^{-1/3}. \tag{32}$$

In Eq. (32) it is revealed that $\bar{J}_{max}$ scales with $Pr^{-1/3}$ and bounded in the domain $0 < \bar{J}_{max} 1.506 Pr^{-1/3}$, when optimization problem actually becomes a non-trivial one.

From the definition (16), it is evident that the parameter $\bar{J}$ represents a competition between convection through the boundary layer and conduction through the insulating material. The value of the parameter $\bar{J}$ of the order of unity signifies a transition between an over all resistance dominated by the insulating material and to that of boundary layer. Thus, it is more realistic to treat the limit $\bar{J} \to 0$ as $\bar{J} \ll 1$ and $\bar{J} \to \infty$ as $\bar{J} \gg 1$. From Eqs. (22) and (28) for non-zero wall thickness it can be read that $\bar{J} \sim 1$. This final result is in qualitative agreement with that obtained in the document [1] after elaborate numerical computations.

6. Insulation with tapered profile

It has been mentioned earlier that Lim et al. [1] assumed a tapered profile of the insulation and numerically solved the convective heat transfer problem with conjugate boundary condition at the top surface of the plate. The authors selected an insulation profile qualitatively similar to an optimum one, as they could not extend calculus of variation for the insulation problem with a rigorous conjugate boundary condition. With the background provided in Section 4 such an assumption is not mandatory for the optimum design of insulation. However, a tapered profile of insulation is still of interest due to the ease of fabrication.

We will show here that analysis presented in Section 4 is general enough to handle the taper profile of the insulation and a closed form expression can be deduced for the minimum heat transport rate. It can be noticed that for $n = 1$ Eq. (28) represents a triangular profile for the distribution of insulating material. The reciprocal of the group $\bar{J}h_L$ is termed as taper parameter. With these Eq. (28) resumes a linearized form

$$\frac{t_{\text{taper}}}{\bar{t}} = 1 + b\left(\frac{1}{2} - \xi\right), \quad \text{where } 0 < b = \frac{1}{\bar{J}h_L} \leqslant 2. \tag{33}$$

The expression for heat transfer rate with this profile is readily obtained from Eq. (21) as

$$\frac{q'_{\text{taper}}}{k_w L \Delta T/\bar{t}} = 2\bar{J}\int_0^1 \frac{\xi \, d\xi}{A\xi^2 + B\xi + C}, \tag{34}$$

where

$$A = -\bar{J}b, \quad B = \left(0.332Pr^{1/3}\right)^{-1} \quad \text{and}$$
$$C = \bar{J}\left(1 + \frac{b}{2}\right). \tag{35}$$

For all possible practical set of values of the parameters

$$4AC - B^2 = -4\bar{J}^2 b\left(1 + \frac{b}{2}\right) - \left(0.332Pr^{1/3}\right)^{-2} < 0. \tag{36}$$

Thus, the algebraic expression for heat transfer rate assumes the form

$$\frac{q'_{\text{taper}}}{k_w L \Delta T/\bar{t}} = 2\bar{J}\ln\left[\left(\frac{A+B+C}{C}\right)^{1/2A} \times \left(\frac{2A + B + \sqrt{B^2 - 4AC}}{2A + B - \sqrt{B^2 - 4AC}} \cdot \frac{B - \sqrt{B^2 - 4AC}}{B + \sqrt{B^2 - 4AC}}\right)^{B/2A\sqrt{B^2-4AC}}\right]. \tag{37}$$

This is the exact solution of the numerical result presented in [1]. Comparing heat transfer results for the representative material volume $\bar{J} = 1$, thermophysical property $Pr = 1$ and the optimum taper parameter $b = 2$, one can verify the relative figure of merit from the ratio

$$\frac{q'_{\text{taper}}}{q'_{\min}} = 1.0095. \tag{38}$$

The last relation reveals that only 0.1% improvement is experienced by the actual optimum profile in place of approximated linearized profile. This taper profile is one such among many other competing designs.

In case of constant wall thickness the taper parameter reduces to zero. The heat transfer can be obtained in algebraic form by evaluating the following reduced integral

$$\frac{q'_{\text{constant}}}{k_w L \Delta T/\bar{t}} = 2\bar{J}\int_0^1 \frac{\xi \, d\xi}{B\xi + \bar{J}}. \tag{39}$$

The final algebraic expression takes the form

$$\frac{q'_{\text{constant}}}{k_w L \Delta T/\bar{t}} = 2\bar{J}\ln\left[e^{1/B}\left(\frac{\bar{J}}{B + \bar{J}}\right)^{\bar{J}/B^2}\right]. \tag{40}$$

Comparative goodness of tapered profile over uniform thickness can be judged by combining the expressions for heat transfer contained in Eqs. (38) and (41) as

$$\frac{q'_{\text{taper}}}{q'_{\text{constant}}} = \ln\left[\left(\frac{A+B+C}{C}\right)^{1/2A}\left(\frac{2A + B + \sqrt{B^2 - 4AC}}{2A + B - \sqrt{B^2 - 4AC}} \cdot \frac{B - \sqrt{B^2 - 4AC}}{B + \sqrt{B^2 - 4AC}}\right)^{B/2A\sqrt{B^2-4AC}}\right] \Big/ \ln\left[e^{1/B}\left(\frac{\bar{J}}{B+\bar{J}}\right)^{\bar{J}/B^2}\right]. \tag{41}$$

It is easy to verify that this ratio is always less than unity for any value of the design parameter b in the bound $[0, 2]$.

7. Commonality of nature of optimization constraints

We will now pay a second look at heuristically obtained constraint that is Eq. (7). Substituting Eq. (9) in Eq. (7) produces an estimate for total resistance, which is exactly the same as that of Eq. (8). Again, Eq. (9) is the general result of variational principle of optimization [1]. Eq. (8) was obtained directly from the material volume constraint, Eq. (2a) and heuristic relation, Eq. (7). This proves the worth of postulating the auxiliary constraint, Eq. (7). The synthesis of this supplementary equation contains the whole physics of the problem. Here, synthetic constraint presupposes that total conductive and convective resistance is "conserved"; though they may not take an equal share at each and every point of the geometry under consideration.

A pertinent example is the Bernoulli equation for a stream tube in an inviscid flow field where, pressure, kinetic and potential energy of the flow compete with each other. The isopotential line provides a basis for understanding the laminar to turbulent transition mechanism as a parallelism between viscid to inviscid transformation [7]. In rigid body mechanics dropping the pressure term we obtain the conservation equation for kinetic and potential energy. At some point of the trajectory the contributing components of a constraint may take an equal share. But this is not necessary condition for optimality (minimum, shortest, quickest, etc.). Existence of isoline is the only rudimental feature of extremality.

In heat transfer literature [8] there is perhaps more misunderstanding than real conflict between power maximization (PM) and entropy generation minimization (EGM) line of optimization. All results obtained otherwise can be reproduced by minimizing the entropy production rate. The concept of isoline can still be invoked in the following manner. Minimizing entropy generation rate $\dot{S}_{\mathrm{gen}}$ with respect to some design variable χ, the first order condition for extrema stands as

$$\frac{\mathrm{d}}{\mathrm{d}\chi}(\dot{S}_{\mathrm{gen}}) = 0. \qquad (42)$$

Thus, for local equilibrium model in some domain of χ we actually have a pseudo constraint

$$\dot{S}_{\mathrm{gen}} = \overline{\dot{S}}_{\mathrm{gen}} \quad (\text{Constant}). \qquad (43)$$

But Eq. (43) constitutes the locus of an isoline and can be deployed with other physical constraints of the model to obtain the condition for optimum. This logical foundation constructs the geometrical interpretation of the optimized results. For an ideally reversible process this constant is identically zero.

After identification of m different competing dissipating mechanisms χ_1, χ_2 and χ_n for a physical process, the synthetic constraint can be laid down as

$$\sum_{i=1}^{m} \chi_m = \overline{\chi}, \qquad (44)$$

where the constant $\overline{\chi}$ is dictated by the finite time and finite resources accessible for a system. For a single contributing mechanism entropy generation between parts of the system can be considered. It has been deduced [9] from purely theoretical reasoning that distribution of driving forces that minimizes the entropy is uniform throughout the system for a single acting irreversibility factor. In literature [9,10] such monotonous distribution of physical or non-physical entities are recognized as principle of equipartition.

8. Conclusions

A heuristic solution methodology for the design problem of thermal insulating systems has been proposed. The technique is an analytical replacement for the formal method of calculus of variation. It is founded on easily perceptible logic and employs a few simple mathematical steps to arrive at the final result.

Closed form expressions for optimum distribution of insulating material for minimum heat transfer from a flat plate when the other side is in convective thermal communication with a forced laminar stream have been obtained. Optimum shape of the profile constitutes an isoline where total resistance contributed by conduction and convection remains uniform throughout the length of the plate. Optimized results are in conformity with the principle of equipartition. Heat transfer results are normalized by the quantity $k_{\mathrm{w}}L\Delta T/\bar{t}$, in which thickness of wall dominates total resistance and provides an effective insulation.

An analytical expression has also been derived for tapered insulation profile. However, for a certain ranges of the parameter [1] the optimum solution exhibits only a marginal improvement over the taper profile. Finally, it goes without saying that any optimization problem plays a meaningful role only when resource is limited. An upper ceiling for the insulating material is prescribed beyond which optimization problem is of no challenge.

References

[1] J.S. Lim, A. Bejan, J.H. Kim, The optimal thickness of a wall with convection on one side, Int. J. Heat Mass Transfer 35 (1992) 1673–1679.

[2] E.R.G. Eckert, R.M. Drake Jr., Analysis of Heat and Mass Transfer, McGraw-Hill Kogakusha Ltd., Tokyo, 1972, p. 312.

[3] D.R. Hartee, On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer, Proc. Cambridge Phil. Soc. 33 (II) (1937) 223–239.

[4] E. Pohlhausen, Der Wämeaustausch zwischen festen Körpern und Flüssigkeiten mit kleiner Reibung und kleiner Wärmeleitung, Z. Angew. Math. Mech. 1 (1921) 115–121.

[5] A. Bejan, Shape and Structure, from Engineering to Nature, Cambridge University Press, UK, 2000, p. 31.

[6] J. Lewins, Bejan's constructal theory of equal potential distribution, Int. J. Heat Mass Transfer 46 (2002) 1541–1543.

[7] A. Bejan, Entropy Generation through Heat and Fluid Flow, Wiley, New York, 1982, pp. 61–62.

[8] A. Bejan, Models of power plants that generate minimum entropy while operating at maximum power, Am. J. Phys. 64 (1996) 1054–1059.

[9] A. Bejan, D. Tondeur, Equipartition, optimal allocation, and the constructal approach to predicting organization in nature, Rev. Gén. Therm. 37 (1998) 165–180.

[10] A. De Vos, B. Desoete, Equipartition principle in finite-time thermodynamics, J. Non-Equilibrium Thermodyn. 25 (2000) 1–13.